Lightcone spin-zero quantum mechanics

1996 J. Phys. A: Math. Gen. 29 6671

(http://iopscience.iop.org/0305-4470/29/20/020)

J. Phys. A: Math. Gen. **29** (1996) 6671–6679. Printed in the UK

Lightcone spin-zero quantum mechanics

S N Mosley
Sunnybank, Albert Rd, Nottingham NG3 4JD, UK

Received 1 July 1996, in final form 13 August 1996

Abstract. We find the eigenfunctions of the Peres spin-zero lightcone energy–momentum operators. These are compared with the usual Klein–Gordon basis states. A continuity equation is found with a positive definite probability density.

1. Lightcone quantum mechanics

In the usual relativistic quantum mechanics we consider the wavefunction $\psi(\boldsymbol{x}, t)$ over constant-time hyperplanes $t = t_1, t = t_2, \ldots$, and given $\psi(\boldsymbol{x}, t)$ the evolution equation enables us to calculate $\psi(\boldsymbol{x}, t + \delta t)$. Dirac [2] long ago suggested an alternative quantum mechanics in which the wavefunction $\psi(\boldsymbol{r}, T)$ is taken over past lightcones $T = T_1, T = T_2, \ldots$ centred on an inertial world line. Here $\boldsymbol{r}$ are the lightcone (retarded) coordinates. The new evolution equation is $\mathrm{i}\,\mathrm{d}\psi(\boldsymbol{r}, T)/\mathrm{d}T = \hat{H}\,\psi(\boldsymbol{r}, T)$, which enables us to calculate the wavefunction on a later past lightcone given $\psi(\boldsymbol{r}, T)$.

One reason for investigating an alternative spin-zero relativistic quantum mechanics is the known problems with the Klein–Gordon theory, in particular that the usual probability density is indefinite (even for positive energy solutions), and that the obvious position operator $\boldsymbol{x}$ is not Hermitian with respect to the usual inner product [3]. Here we will find a positive-definite probability density, and also the position operator $\boldsymbol{r}$ is Hermitian with respect to the inner product (2.1). Additional advantages of the lightcone formalism are: (i) Lorentz transformations are easier to describe in that a past lightcone is mapped onto itself, (ii) retarded particle interactions are natural to the lightcone formalism, (iii) the projection postulate is more plausible in the case of the past lightcone wavefunction [4]. On the other hand the lightcone energy–momentum operators contain non-local (integral) operators.

Peres [1] found a set of spin-zero Poincaré group operators acting on the past lightcone. We will present his energy–momentum operators in a more tractable form, and find their eigenfunctions. We also find a continuity equation resulting from the Hamiltonian.

The question naturally arises as to whether we are simply performing a coordinate transformation, in which case no new physics arises. We will compare the time-dependant wavefunctions of definite momentum found here with the corresponding Klein–Gordon wavefunctions, and discuss their differences. The author of this paper has co-authored a previous paper [5] in which a past lightcone quantum mechanics was considered, the theory therein suffers from the same defects as the Klein–Gordon theory—in particular an indefinite probability density.

The past lightcone with vertex at the origin is parametrized by the null 4-vector $r^\lambda \equiv (-r, \boldsymbol{r})$, with $\boldsymbol{r}$ the retarded position coordinate. The Lorentz generators map the past lightcone onto itself, and defining the classical Lorentz generators by

$$\boldsymbol{K} \equiv (J^{10}, J^{20}, J^{30}) = r\,\boldsymbol{\pi} \qquad \boldsymbol{J} \equiv (J^{23}, J^{31}, J^{12}) = \boldsymbol{r} \times \boldsymbol{\pi} \tag{1.1}$$

where $\boldsymbol{\pi}$ is the conjugate momentum (such that the Poisson bracket $\{r^a\,,\,\pi^b\} = \delta^{ab}$), we can check that

$$\{J^{\lambda\mu}, r^\nu\} = \eta^{\mu\nu} r^\lambda - \eta^{\lambda\nu} r^\mu\,. \tag{1.2}$$

where the metric tensor $\eta^{\mu\nu}$ is diagonal with $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = 1$. Any other vector satisfying the relation equivalent to (1.2) we define to be a 4-vector, i.e. covariant under Lorentz transformations.

Next we consider the lightcone energy–momentum generators for a single free particle of mass m, which are [1, 2, 6]

$$H \equiv P^0 = \frac{1}{2} r \frac{\boldsymbol{\pi}^2 + m^2}{\boldsymbol{r}\cdot\boldsymbol{\pi}} \qquad \boldsymbol{P} = \boldsymbol{\pi} - \frac{1}{2}\boldsymbol{r}\frac{\boldsymbol{\pi}^2 + m^2}{\boldsymbol{r}\cdot\boldsymbol{\pi}} = \boldsymbol{\pi} - \hat{\boldsymbol{r}}\, P^0\,. \tag{1.3}$$

It may be verified that $P^{0^2} = m^2 + \boldsymbol{P}^2$. The reason for the more complicated form of the lightcone momentum generators (1.3) is that a space translated past lightcone in general meets the particle at a different point on its world line, so that there is an additional convection term due to particle motion. Thus the Poisson bracket relation between P and r is

$$\{P^\lambda\,,\, r^\mu\} = \eta^{\lambda\mu} - \frac{r^\lambda P^\mu}{r\cdot P} \tag{1.4}$$

as may be verified. The relation (1.4) arises in the theory of retarded potentials, see for example equations (40.10)–(40.11) in [7]. Altogether the generators (1.1), (1.3) satisfy the Poisson bracket relations characteristic of the Poincaré group.

Note that the Hamiltonian P^0 of (1.3) is not obviously positive-definite, but depends on the sign of the $\boldsymbol{r}\cdot\boldsymbol{\pi}$ term in the denominator. From equation (1.3) we can verify that

$$\boldsymbol{r}\cdot\boldsymbol{\pi} = rP^0 + \boldsymbol{r}\cdot\boldsymbol{P} = r\sqrt{m^2 + \boldsymbol{P}^2} + \boldsymbol{r}\cdot\boldsymbol{P}\,. \tag{1.5}$$

So if $P^0 = +\sqrt{m^2 + \boldsymbol{P}^2}$, then from the Schwarz inequality it follows that $\boldsymbol{r}\cdot\boldsymbol{\pi} > 0$ for a massive particle, similarly if $P^0 = -\sqrt{m^2 + \boldsymbol{P}^2}$ then $\boldsymbol{r}\cdot\boldsymbol{\pi} < 0$. In the classical (non-quantized) case we always take the positive square root in which case $\boldsymbol{r}\cdot\boldsymbol{\pi}$ is positive. When in the next section we quantize the P^λ generators to form the corresponding operators p^λ, the operator identity $p^{0^2} = m^2 + \boldsymbol{p}^2$ holds, but the operator p^0 can also have negative eigenvalues, i.e. there are negative energy solutions. This implies that the operator corresponding to $\boldsymbol{r}\cdot\boldsymbol{\pi}$ can also have negative eigenvalues.

2. Quantization

We require the operators $j^{\lambda\mu}$, p^ν corresponding to the classical generators $J^{\lambda\mu}$, P^ν to be symmetric with respect to $\mathcal{H}_r$, which is the Lorentz-invariant positive-definite scalar product space over the past lightcone

$$\mathcal{H}_r: \qquad \left\langle\phi\left|\frac{1}{r}\right|\psi\right\rangle \equiv \int \phi^*(\boldsymbol{r})\frac{1}{r}\psi(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}\,. \tag{2.1}$$

The expectation value of any operator $\mathcal{O}$ is then $\langle\psi|(1/r)|\mathcal{O}\psi\rangle$ with $\langle\psi|(1/r)|\psi\rangle = 1$. Note that the position operator, multipication by $\boldsymbol{r}$, is evidently symmetric in $\mathcal{H}_r$. Also the operators $j^{\lambda\mu}$, p^{ν} must obey the Poincaré group commutation relations. We will find the eigenfunctions of the p^{λ} operators and check their orthogonality relations.

The Lorentz generators (1.1) are readily quantized [1, 6]:

$$\boldsymbol{j} \equiv (j^{23}, j^{31}, j^{12}) = -\mathrm{i}\boldsymbol{r} \times \boldsymbol{\nabla} \qquad \boldsymbol{k} \equiv (j^{10}, j^{20}, j^{30}) = -\mathrm{i}r\,\boldsymbol{\nabla}. \tag{2.2}$$

We tentatively identify the operator corresponding to $\boldsymbol{r}\cdot\boldsymbol{\pi}$ as the dilation operator $-\mathrm{i}\partial_r r \equiv \Sigma$. The left and right inverse of Σ is Σ^{-1}, defined [5] as

$$\Sigma^{-1} f(y,\theta,\phi) = \frac{\mathrm{i}}{y}\int_0^y f(y',\theta,\phi)\,\mathrm{d}y' \qquad \text{equivalently} \qquad \Sigma^{-1} f(\boldsymbol{r}) = \mathrm{i}\int_0^1 f(\alpha\boldsymbol{r})\,\mathrm{d}\alpha. \tag{2.3}$$

The operators Σ , Σ^{-1} are Lorentz invariant, i.e. commute with $j^{\lambda\mu}$. The generators P^{λ} of (1.1) may be written $P^{\lambda} = (A^{\lambda} - \frac{1}{2}m^2 r^{\lambda})/(\boldsymbol{r}\cdot\boldsymbol{\pi})$ where [8]

$$A^{\lambda} \equiv (A^0\,,\ \boldsymbol{A}) = (\tfrac{1}{2}r\boldsymbol{\pi}^2\,,\ \boldsymbol{\pi}(\boldsymbol{r}\cdot\boldsymbol{\pi}) - \tfrac{1}{2}\boldsymbol{r}\boldsymbol{\pi}^2\,) \tag{2.4}$$

is the 4-vector version of the Runge–Lenz vector. The corresponding operators [8] are

$$a^{\lambda} \equiv (a^0\,,\ \boldsymbol{a}) = (-\tfrac{1}{2}r\boldsymbol{\nabla}^2\,,\ -\mathrm{i}\Sigma\,\boldsymbol{\nabla} + \tfrac{1}{2}\boldsymbol{r}\boldsymbol{\nabla}^2\,) \tag{2.5}$$

which are symmetric in $\mathcal{H}_r$. These a^{λ} have the following properties:

$$[a^{\lambda}\,,\ a^{\mu}] = 0 \tag{2.6}$$

$$[a^{\lambda}\,,\ r^{\mu}] = \mathrm{i}\Sigma\eta^{\lambda\mu} + \mathrm{i}\,j^{\lambda\mu} \tag{2.7}$$

$$a^{\lambda}\cdot a_{\lambda} = 0 \qquad r^{\lambda}\cdot a_{\lambda} = -(\Sigma+\mathrm{i})^2 \qquad a^{\lambda}\cdot r_{\lambda} = -(\Sigma-\mathrm{i})^2 \tag{2.8}$$

$$(a^{\lambda}r^{\mu} - a^{\mu}r^{\lambda}) = -(\Sigma-\mathrm{i})\,j^{\lambda\mu} = -\left(\frac{1}{r}\Sigma r\right) j^{\lambda\mu} \tag{2.9}$$

$$(r^{\lambda}a^{\mu} - r^{\mu}a^{\lambda}) = (\Sigma+\mathrm{i})\,j^{\lambda\mu} = \left(r\Sigma\frac{1}{r}\right) j^{\lambda\mu}\,. \tag{2.10}$$

It is interesting to note that the operators $(a^{\lambda} + r^{\lambda}/2)$, $(a^{\lambda} - r^{\lambda}/2)$, $j^{\lambda\mu}$, Σ together form a realization of the O(4, 2) algebra [8].

There are a number of ways of ordering the operators within p^{λ}, corresponding to the classical generators P^{λ} (1.3), but the requirements that $[p^{\lambda}\,,\ p^{\mu}] = 0$, and that the p^{λ} are symmetric in $\mathcal{H}_r$, lead to the choice

$$p^{\lambda} = \frac{1}{\sqrt{r}}\Sigma^{-1}\sqrt{r}\,a^{\lambda} - \frac{1}{4}m^2(r^{\lambda}\Sigma^{-1} + \Sigma^{-1}r^{\lambda}) \tag{2.11}$$

so that the evolution equation is

$$\mathrm{i}\,\frac{\mathrm{d}\psi}{\mathrm{d}T} = p^0\,\psi = \left[-\frac{1}{2}\left(\frac{1}{\sqrt{r}}\Sigma^{-1}\sqrt{r}\right)r\boldsymbol{\nabla}^2 + \frac{1}{4}m^2(r\Sigma^{-1} + \Sigma^{-1}r)\right]\psi\,. \tag{2.12}$$

The ten operators (2.2) and (2.11) are equivalent to those found by Peres [1]. In the case of the energy–momentum operators the equivalence is non-trivial and will be demonstrated in appendix C.

To prove the required commutation relations $\left[p^\lambda\,,\ p^\mu\right] = 0$ appears a formidable task, but computations are greatly eased using the fact that Σ or Σ^{-1} commute with any operator $\mathcal{O}$ homogeneous of degree zero (i.e. such that $\mathcal{O}(\alpha\boldsymbol{r}) = \mathcal{O}(\boldsymbol{r})$). Then operators such as $\sqrt{r}\,a^\lambda$ (homogeneous of degree $-\frac{1}{2}$) may be reordered with Σ^{-1} as follows:

$$\Sigma^{-1}\sqrt{r}\,a^\lambda = \Sigma^{-1}(\sqrt{r}\,a^\lambda\sqrt{r})\frac{1}{\sqrt{r}} = \sqrt{r}\,a^\lambda\sqrt{r}\,\Sigma^{-1}\frac{1}{\sqrt{r}}\,. \tag{2.13}$$

Then with equations (2.9) and (2.10) we can prove $\left[p^\lambda\,,\ p^\mu\right] = 0$, which will be shown in appendix A.

Our approach to quantizating the P^λ generators has been formal in the sense that we have been led by the requirements that the p^λ operators are (i) symmetric in $\mathcal{H}_r$, and (ii) commute with each other. Recently two papers [9, 10] have suggested that quantization of the P^λ generators must be carried out under the constraint $\boldsymbol{r}\cdot\boldsymbol{\pi} > 0$. However, as we have noted at the end of section 1 $\boldsymbol{r}\cdot\boldsymbol{\pi}$ is only positive when $P^0 = +\sqrt{m^2 + \boldsymbol{P}^2}$, but negative when $P^0 = -\sqrt{m^2 + \boldsymbol{P}^2}$. In the next section we will see that there are indeed negative energy solutions.

2.1. The continuity equation

It follows from the Hamiltonian (2.12) that we can derive a continuity equation in the form

$$\frac{\partial}{\partial T}\rho + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0 \tag{2.14}$$

where

$$\rho = \psi^*\frac{1}{r}\psi \tag{2.15}$$

$$\boldsymbol{J} = \frac{1}{2}\left(-\mathrm{i}\psi^*\boldsymbol{\nabla}\sqrt{r}\,\Sigma^{-1}\frac{1}{\sqrt{r}}\psi + \frac{1}{2}\hat{\boldsymbol{r}}(\Sigma^{-1}\sqrt{r}\,\boldsymbol{\nabla}\psi^*\cdot\Sigma^{-1}\sqrt{r}\boldsymbol{\nabla}\psi + m^2\Sigma^{-1}\psi^*\Sigma^{-1}r\psi)\right) + \mathrm{CC} \tag{2.16}$$

where CC stands for the complex conjugate terms. The proof we leave to appendix B.

3. The energy–momentum eigenfunctions

Call the eigenvalues of the p^λ energy–momentum operators $k^\lambda \equiv (k^0, \boldsymbol{k})$, with $k^0 \equiv \sqrt{m^2 + \boldsymbol{k}^2}$. We expect eigenfunctions of these operators to be functions of the positive-definite Lorentz scalar $\zeta \equiv -k^\lambda r_\lambda = k^0 r + \boldsymbol{k}\cdot\boldsymbol{r}$. We will show that

$$\phi_{\boldsymbol{k}}(\boldsymbol{r}) = \left[(1 + \mathrm{i}\zeta)J_0(\zeta/2) - \zeta J_1(\zeta/2)\right]\mathrm{e}^{\mathrm{i}\zeta/2} \tag{3.1}$$

are the required eigenfunctions. $\phi_{\boldsymbol{k}}(\boldsymbol{r})$ satisfies

$$\zeta\phi'' + (1 - i\zeta)\phi' - \tfrac{3}{2}\mathrm{i}\phi = 0 \tag{3.2}$$

(ϕ' denoting differentiation of ϕ with respect to the argument) as may be verified directly. So ϕ is a confluent hypergeometric function with imaginary argument, i.e. $\phi \equiv {}_1F_1(\frac{3}{2}, 1, \mathrm{i}\,\zeta)$. Equation (3.2) may be written

$$\mathrm{i}\Sigma\phi' + \frac{1}{\sqrt{\zeta}}\Sigma\sqrt{\zeta}\phi = 0 \tag{3.3}$$

as $\Sigma \equiv -\mathrm{i}\partial_r r = -\mathrm{i}\partial_\zeta \zeta$. From equation (3.3) it follows that

$$\phi' = \mathrm{i}\Sigma^{-1}\frac{1}{\sqrt{r}}\Sigma\sqrt{r}\phi = \mathrm{i}\frac{1}{\sqrt{r}}\Sigma\sqrt{r}\Sigma^{-1}\phi\,. \tag{3.4}$$

To show that $p^0\phi = k^0\phi$ we first need

$$\begin{aligned} a^0\phi \equiv -\tfrac{1}{2}r\nabla^2\phi &= -\tfrac{1}{2}r\,\nabla\cdot[(k^0\hat{\boldsymbol{r}} + \boldsymbol{k})\phi'] \\ &= -k^0\phi' - \tfrac{1}{2}r(k^{0^2} + 2k^0\hat{\boldsymbol{r}}\cdot\boldsymbol{k} + \boldsymbol{k}^2)\phi'' \\ &= -k^0(\phi' + \zeta\phi'') + \tfrac{1}{2}m^2 r\phi'' \\ &= k^0\left(\frac{1}{\sqrt{r}}\Sigma\sqrt{r}\phi\right) - \frac{1}{2}m^2\left(\frac{r}{\zeta}\right)\left(\phi' + \frac{1}{\sqrt{r}}\Sigma\sqrt{r}\phi\right) \end{aligned} \tag{3.5}$$

using equation (3.2). Then

$$\begin{aligned} p^0\,\phi &\equiv \left(\frac{1}{\sqrt{r}}\Sigma^{-1}\sqrt{r}\right)a^0\phi + \frac{1}{4}m^2(r\Sigma^{-1} + \Sigma^{-1}r)\,\phi \\ &= k^0\phi - \frac{1}{2}m^2\left(\frac{r}{\zeta}\right)\left[\frac{1}{\sqrt{r}}\Sigma^{-1}\sqrt{r}\phi' + \phi\right] + \frac{1}{4}m^2(r\Sigma^{-1} + \Sigma^{-1}r)\,\phi \\ &= k^0\phi - \frac{1}{2}m^2\left(\frac{r}{\zeta}\right)\left[\mathrm{i}\Sigma^{-1}\phi + \phi\right] + \frac{1}{4}m^2(r\Sigma^{-1} + \Sigma^{-1}r)\,\phi \end{aligned} \tag{3.6}$$

where we have used equation (3.4). The m^2 terms in (3.6) may be shown to cancel with the aid of the following identities:

$$\Sigma^{-1}\phi = \left[\mathrm{i}J_0(\zeta/2) - J_1(\zeta/2)\right]\mathrm{e}^{\mathrm{i}\zeta/2} \tag{3.7}$$

$$\Sigma^{-1}r\phi = r\left[\mathrm{i}J_0(\zeta/2) - \left(1 + \frac{2\mathrm{i}}{\zeta}\right)J_1(\zeta/2)\right]\mathrm{e}^{\mathrm{i}\zeta/2}\,. \tag{3.8}$$

So finally $p^0\phi = k^0\phi$.

Next consider $\boldsymbol{p}\phi$:

$$\begin{aligned} \boldsymbol{p}\phi &= \left[-\mathrm{i}\left(\frac{1}{\sqrt{r}}\Sigma^{-1}\sqrt{r}\right)\Sigma\boldsymbol{\nabla} - \hat{\boldsymbol{r}}p^0\right]\phi \\ &= -\mathrm{i}\left(\frac{1}{\sqrt{r}}\Sigma^{-1}\sqrt{r}\right)\Sigma(\hat{\boldsymbol{r}}k^0 + \boldsymbol{k})\phi' - \hat{\boldsymbol{r}}k^0\phi \\ &= -\mathrm{i}\left(\frac{1}{\sqrt{r}}\Sigma^{-1}\sqrt{r}\right)\Sigma(\hat{\boldsymbol{r}}k^0 + \boldsymbol{k})\left(\mathrm{i}\frac{1}{\sqrt{\zeta}}\Sigma\sqrt{\zeta}\Sigma^{-1}\phi\right) - \hat{\boldsymbol{r}}k^0\phi = \boldsymbol{k}\phi \end{aligned} \tag{3.10}$$

where we have used equation (3.4).

The function $\phi_{\boldsymbol{k}}(\boldsymbol{r})$ was found by Derrick [6] who considered a variety of basis states orthogonal in $\mathcal{H}_r$, (see his equation (B8) therein). Derrick showed that

$$\left\langle\phi_{\boldsymbol{k}}\left|\frac{1}{r}\right|\phi_{\boldsymbol{k}'}\right\rangle = (2^5\pi^2)k^0\delta(\boldsymbol{k} - \boldsymbol{k}')\,. \tag{3.11}$$

3.1. Comparison with the Klein–Gordon wavefunctions

Consider the time-dependant wavefunction $\psi(\boldsymbol{r}, T)$ of momentum $\boldsymbol{k}$, which is

$$\psi(\boldsymbol{r}, T) = \mathrm{e}^{-\mathrm{i}k^0 T}\phi_{\boldsymbol{k}}(\boldsymbol{r}) \equiv \mathrm{e}^{-\mathrm{i}k^0 T + \mathrm{i}\zeta/2}\big[J_0(\zeta/2 + \mathrm{i}\zeta\, J_0(\zeta/2) - \zeta\, J_1(\zeta/2)\big] \tag{3.12}$$

with $\zeta \equiv k^0 r + \boldsymbol{k}\cdot\boldsymbol{r}$. The first term falls off rapidly away from the origin, and omitting this term for the far field

$$\begin{aligned}\psi(\boldsymbol{r}, T) &\simeq \mathrm{i}\zeta\,\mathrm{e}^{-\mathrm{i}k^0 T + \mathrm{i}\zeta/2}\big[J_0(\zeta/2) + \mathrm{i}J_1(\zeta/2)\big] \\ &\to 2\mathrm{i}\sqrt{\frac{\zeta}{\pi}}\,\mathrm{e}^{-\mathrm{i}\pi/4}\,\mathrm{e}^{-\mathrm{i}k^0 T + \mathrm{i}\zeta} \qquad \text{as} \quad \zeta \to \infty\,.\end{aligned} \tag{3.13}$$

The exponential $\mathrm{e}^{-\mathrm{i}k^0 T + \mathrm{i}\zeta} \equiv \mathrm{e}^{-\mathrm{i}k^0 T + \mathrm{i}k^0 r + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$, and after performing the coordinate transformation $T - r = t$, $\boldsymbol{r} = \boldsymbol{x}$, is equivalent to the Klein–Gordon eigenstate $\mathrm{e}^{-\mathrm{i}k^0 t + i\boldsymbol{k}\cdot\boldsymbol{x}}$. The $\sqrt{\zeta}$ factor in (3.13) is partly accounted for by the fact that our probability density is $((1/r)\psi^*\psi)$; however, this still leaves the angular variable $\sqrt{\zeta/r}$ as a factor, i.e. the asymptotic lightcone wavefunction $\phi_{\boldsymbol{k}}(\boldsymbol{r})$ is of greater amplitude in the direction $\boldsymbol{k}$ than in the direction $-\boldsymbol{k}$, in contrast to the Klein–Gordon case. The classical motion of a free particle in lightcone coordinates is also asymmetric about the origin, in that the apparent velocity of the particle when approaching the origin is greater than when it is receding.

3.2. Negative energy solutions

We note that p^0 is a purely imaginary operator. This means that $p^0\phi^*_{\boldsymbol{k}} = -k^0\phi^*_{\boldsymbol{k}}$. Thus the general solution of the evolution equation (2.12), including the positive and negative energy solutions $\psi_{(+)}$ and $\psi_{(-)}$, is

$$\psi = \psi_{(+)} + \psi_{(-)} \equiv \int \mathrm{e}^{-\mathrm{i}k^0 T}\,\alpha(\boldsymbol{k})\,\phi_{\boldsymbol{k}}(\boldsymbol{r})\;\mathrm{d}^3 k + \int \mathrm{e}^{\mathrm{i}k^0 T}\,\beta(\boldsymbol{k})\,\phi^*_{\boldsymbol{k}}(\boldsymbol{r})\;\mathrm{d}^3 k \tag{3.14}$$

with $k^0 = \sqrt{m^2 + \boldsymbol{k}^2}$, and $\alpha(\boldsymbol{k})$, $\beta(\boldsymbol{k})$ being arbitrary functions of $\boldsymbol{k}$.

4. Conclusion and outlook

It appears that the spin-zero lightcone quantum mechanics considered here has some advantages, particularly a positive-definite probability density, and a Hermitian position operator $\boldsymbol{r}$. The basis states are not simply a coordinate transformation from the Klein–Gordon basis states. We have not extended the theory to include interaction terms here; we hope to address this elsewhere.

Appendix A. On the commutator $[p^\lambda, p^\mu]$

The commutator $[p^\lambda, p^\mu]$ contains m^4, m^2, and massless terms. The m^4 terms in the commutator are readily shown to be zero using the fact that $\hat{\boldsymbol{r}}$ commutes with Σ^{-1}. Similarly the massless terms are zero using equation (2.6). The m^2 terms of the commutator $[p^\lambda\,,\;p^\mu]$ are

$$-\frac{m^2}{4}\Bigg[\Big(\frac{1}{\sqrt{r}}\Sigma^{-1}\sqrt{r}\Big)\Big(a^\lambda r^\mu - a^\mu r^\lambda\Big)\frac{1}{r}(r\Sigma^{-1}+\Sigma^{-1}r) + (r\Sigma^{-1}+\Sigma^{-1}r)\frac{1}{r}(r^\lambda a^\mu - r^\mu a^\lambda)\Big(\sqrt{r}\Sigma^{-1}\frac{1}{\sqrt{r}}\Big)\Bigg]. \qquad (A.1)$$

Then inserting the identities (2.9) and (2.10), which are

$$(a^\lambda r^\mu - a^\mu r^\lambda) = -\Big(\frac{1}{r}\Sigma r\Big)j^{\lambda\mu} \qquad (r^\lambda a^\mu - r^\mu a^\lambda) = \Big(r\Sigma\frac{1}{r}\Big)j^{\lambda\mu}$$

and noting that

$$\begin{aligned}(r\Sigma^{-1}+\Sigma^{-1}r) &= \Sigma^{-1}(r\Sigma+\Sigma r)\Sigma^{-1}\\ &= 2\,\Sigma^{-1}(\sqrt{r}\Sigma\sqrt{r})\Sigma^{-1}\end{aligned} \qquad (A.2)$$

then equation (A.1) becomes

$$\begin{aligned}&-\frac{m^2}{2}\Bigg[-\Big(\frac{1}{\sqrt{r}}\Sigma^{-1}\sqrt{r}\Big)\Big(\frac{1}{r}\Sigma r\Big)j^{\lambda\mu}\frac{1}{r}(\Sigma^{-1}\sqrt{r}\Sigma\sqrt{r}\Sigma^{-1})\\ &\qquad + (\Sigma^{-1}\sqrt{r}\Sigma\sqrt{r}\Sigma^{-1})\frac{1}{r}\Big(r\Sigma\frac{1}{r}\Big)j^{\lambda\mu}\Big(\sqrt{r}\Sigma^{-1}\frac{1}{\sqrt{r}}\Big)\Bigg]\\ &= -\frac{m^2}{2}\big[-j^{\lambda\mu}\,\Sigma^{-1} + j^{\lambda\mu}\,\Sigma^{-1}\big] = 0\,.\end{aligned} \qquad (A.3)$$

Appendix B. On the continuity equation (2.14)

We calculate $\boldsymbol{\nabla}\cdot\boldsymbol{J}$ using the identity $\boldsymbol{\nabla}\cdot(\hat{\boldsymbol{r}}\,u\,v) = (\mathrm{i}/r)u(\Sigma v) + (\mathrm{i}/r)v(\Sigma u)$ for any functions u, v. In what follows CC stands for complex conjugate terms:

$$\begin{aligned}2(\boldsymbol{\nabla}\cdot\boldsymbol{J}) &\equiv \boldsymbol{\nabla}\cdot\Bigg(-\mathrm{i}\psi^*\,\boldsymbol{\nabla}\sqrt{r}\Sigma^{-1}\frac{1}{\sqrt{r}}\psi + \frac{m^2}{2}\,\hat{\boldsymbol{r}}(\Sigma^{-1}\sqrt{r}\boldsymbol{\nabla}\psi^*\cdot\Sigma^{-1}\sqrt{r}\boldsymbol{\nabla}\psi)\\ &\qquad + \Sigma^{-1}\psi^*\,\Sigma^{-1}r\psi\Bigg) + \mathrm{CC}\\ &= -\mathrm{i}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\sqrt{r}\Sigma^{-1}\frac{1}{\sqrt{r}}\psi - \mathrm{i}\psi^*\cdot\boldsymbol{\nabla}^2\sqrt{r}\Sigma^{-1}\frac{1}{\sqrt{r}}\psi\\ &\qquad + \frac{m^2}{2}\Bigg[\Big(\frac{\mathrm{i}}{\sqrt{r}}\boldsymbol{\nabla}\psi^*\Big)\cdot\Sigma^{-1}\sqrt{r}\boldsymbol{\nabla}\psi + \Sigma^{-1}\sqrt{r}\boldsymbol{\nabla}\psi^*\cdot\Big(\frac{\mathrm{i}}{\sqrt{r}}\boldsymbol{\nabla}\psi\Big)\\ &\qquad + \frac{\mathrm{i}}{r}\psi^*\,\Sigma^{-1}r\psi + \Sigma^{-1}\psi^*\frac{\mathrm{i}}{r}r\psi\Bigg] + \mathrm{CC}\end{aligned}$$

$$= -\mathrm{i}\psi^* \boldsymbol{\nabla}^2\sqrt{r}\,\Sigma^{-1}\frac{1}{\sqrt{r}}\psi - \mathrm{i}\boldsymbol{\nabla}^2\sqrt{r}\,\Sigma^{-1}\frac{1}{\sqrt{r}}\psi^*\psi$$
$$+ \frac{m^2}{2}\left[\frac{\mathrm{i}}{r}\psi^*\Sigma^{-1}r\psi + \Sigma^{-1}\psi^*\frac{\mathrm{i}}{r}r\psi + \Sigma^{-1}r\psi^*\frac{\mathrm{i}}{r}\psi + \frac{\mathrm{i}}{r}r\psi^*\Sigma^{-1}\psi\right]$$
$$= -2\psi^*\frac{1}{r}\frac{\partial}{\partial T}\psi - 2\left(\frac{1}{r}\frac{\partial}{\partial T}\psi^*\right)\psi = -2\frac{\partial}{\partial T}\rho \qquad \text{(B.1)}$$

as required.

Appendix C. The Peres spin-zero operators [1]

The spin-zero Peres Hamiltonian, correcting a misprint in equation (28) from [1], is

$$H = \frac{1}{2}\left(p + \frac{m^2}{p + \frac{1}{4}(rpr)^{-1}} + \frac{\boldsymbol{J}^2 + \frac{1}{4}}{rpr}\right) \qquad \text{(C.1)}$$

where Peres' p in our notation is $p \equiv (1/\sqrt{r})\Sigma(1/\sqrt{r})$. Then equation (C.1) is

$$p^0 = \frac{1}{2}\left(\frac{1}{\sqrt{r}}\Sigma\frac{1}{\sqrt{r}} + \frac{m^2}{\left[\frac{1}{\sqrt{r}}\Sigma\frac{1}{\sqrt{r}} + \frac{1}{4}\frac{1}{\sqrt{r}}\Sigma^{-1}\frac{1}{\sqrt{r}}\right]} + \frac{1}{\sqrt{r}}\Sigma^{-1}\frac{1}{\sqrt{r}}\left(\boldsymbol{J}^2 + \frac{1}{4}\right)\right)$$
$$= \frac{1}{2}\left(\left[\frac{1}{\sqrt{r}}\Sigma\frac{1}{\sqrt{r}} + \frac{1}{4}\frac{1}{\sqrt{r}}\Sigma^{-1}\frac{1}{\sqrt{r}}\right] + m^2\left[\frac{1}{\sqrt{r}}\Sigma\frac{1}{\sqrt{r}} + \frac{1}{4}\frac{1}{\sqrt{r}}\Sigma^{-1}\frac{1}{\sqrt{r}}\right]^{-1}\right.$$
$$\left. + \frac{1}{\sqrt{r}}\Sigma^{-1}\frac{1}{\sqrt{r}}\boldsymbol{J}^2\right). \qquad \text{(C.2)}$$

The operator in square brackets is invertible noting that

$$\left(\Sigma\frac{1}{\sqrt{r}}\right)\Sigma^{-1}\left(\frac{1}{\sqrt{r}}\Sigma\right) = \left(\frac{1}{\sqrt{r}}\Sigma + \frac{\mathrm{i}}{2\sqrt{r}}\right)\Sigma^{-1}\left(\Sigma\frac{1}{\sqrt{r}} - \frac{\mathrm{i}}{2\sqrt{r}}\right)$$
$$= \left[\frac{1}{\sqrt{r}}\Sigma\frac{1}{\sqrt{r}} + \frac{1}{4}\frac{1}{\sqrt{r}}\Sigma^{-1}\frac{1}{\sqrt{r}}\right] \qquad \text{(C.3)}$$

implying that

$$\left[\frac{1}{\sqrt{r}}\Sigma\frac{1}{\sqrt{r}} + \frac{1}{4}\frac{1}{\sqrt{r}}\Sigma^{-1}\frac{1}{\sqrt{r}}\right]^{-1} = \Sigma^{-1}\sqrt{r}\,\Sigma\sqrt{r}\,\Sigma^{-1}. \qquad \text{(C.4)}$$

So equation (C.2) is

$$p^0 = \frac{1}{2}\left(\Sigma\frac{1}{\sqrt{r}}\Sigma^{-1}\frac{1}{\sqrt{r}}\Sigma + m^2\Sigma^{-1}\sqrt{r}\,\Sigma\sqrt{r}\,\Sigma^{-1} + \frac{1}{\sqrt{r}}\Sigma^{-1}\frac{1}{\sqrt{r}}\boldsymbol{J}^2\right)$$
$$= \frac{1}{2}\left(\frac{1}{\sqrt{r}}\Sigma^{-1}\sqrt{r}\,\Sigma\frac{1}{r}\Sigma + \frac{m^2}{2}(r\Sigma^{-1} + \Sigma^{-1}r) + \frac{1}{\sqrt{r}}\Sigma^{-1}\frac{1}{\sqrt{r}}\boldsymbol{J}^2\right) \qquad \text{(C.5)}$$

recalling (A.2). Then substituting $\boldsymbol{J}^2 = -r^2\nabla^2 - r\Sigma(1/r)\Sigma$ we see that equation (C.5) is equivalent to the p^0 of (2.12).

References

[1] Peres A 1967 *J. Math. Phys.* **9** 785
[2] Dirac P A M 1949 *Rev. Mod. Phys.* **21** 392
[3] Newton T D and Wigner E P 1949 *Rev. Mod. Phys.* **21** 400
[4] Finkelstein J 1994 *Phys. Lett.* **188A** 117
[5] Mosley S N and Farina J E G 1992 *J. Phys. A: Math. Gen.* **25** 4673
[6] Derrick G H 1987 *J. Math. Phys.* **28** 1327
[7] Rindler W 1991 *Introduction to Special Relativity* (Oxford: Clarendon) 2nd edn
[8] Barut A O and Bornzin G L 1971 J. Math. Phys.**12** 841
[9] Wan K K and Powis J J 1994 *Int. J. Theor. Phys.* **33** 553
[10] Levin O and Peres A 1994 *J. Phys. A: Math. Gen.* **27** L143